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BOUNDARY-VALUE PROBLEM FOR THE KINETIC EQUATION IN A LAYER WITH MIRROR BOUNDARY CONDITIONS

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Introduction. The explicit distribution function of a rarefied gas in a layer with mirror boundary conditions has been constructed, the lower plate that bound the gas executing normal harmonic oscillations and the upper one being at rest. The Boltzmann unsteady-state equation with a collision operation in Bhatnagar–Gross–Crook (BGC) form has been applied.

To find the explicit distribution function, we use the expansion method for solving boundary-value problems in generalized eigenfunctions of an appropriate characteristic equation. The fundamentals of this method were given by Case and Zweifel [1]. As in [1], attempts at analytical solutions of the kinetic equation in the layer for critical-layer problems in the theory of nuclear reactors and the Couette and Poiseuille and other problems were also reviewed by Greenberg et al. [2] and Cercignani [3, 4]. Numerical-analytical methods were employed in all the papers mentioned above.

In the present paper, we derive the exact solution of the boundary-value problem for the kinetic equation in the layer, with modification of the Case–Zweifel method [1]. Note that precisely the modification of this method made it possible to solve [5–8] a number of problems for model kinetic equations that had been insoluble for a long time. Among such problems there are the problem of temperature-jump calculation [5], the Landau problem (which was solved by him exactly for a half-space) of the electron-plasma behavior in the layer [6], and the problem of strong evaporation for a one-dimensional [7] or three-dimensional [8] gas.

Note that Aoki and Cercignani [9, 10] attempted to develop the Case–Zweifel method for exact solution of the haft-spatial boundary-value problem for the Boltzmann model unsteady-state equation. However, the theory based on the technique of Abelian differentials on Riemann surfaces is so complicated that it has so far not been used for solution of applied problems. Note for comparison that the method that is being developed in the present paper allows one to construct the distribution function of a rarefied gas in the layer in closed form.

1. Formulation of the Problem. We consider a layer of thickness d, which is filled by a rarefied gas. The lower plate which bounds the gas lies in the plane $x = 0$, while the upper plate lies in the plane $x = d$. The x axis is perpendicular to the plates. The lower plate executes normal harmonic oscillations with frequency ω and amplitude U ($x = Ue^{i\omega t}$) relative to its equilibrium position ($x = 0$). The upper plate is fixed rigidly. We have to construct the distribution function of the gas molecules.

We use the Boltzmann model kinetic equation with a collision operator in BGC form (see, e.g., [11]):

$$\left(\frac{\partial}{\partial t} + \mu\,\frac{\partial}{\partial x} + 1\right) Y(t, x, \mu) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp(-\mu'^2) Y(t, x, \mu')\, d\mu', \tag{1.1}$$

where μ is the projection of the molecular velocity on the x axis. The boundary conditions are obtained from the problem condition:

$$\begin{aligned} Y(t,0,\mu) &= Y(t,0,-\mu) + 2U\mu e^{i\omega t}, \qquad t > 0, \qquad \mu > 0; \\ Y(t,d,\mu) &= Y(t,d,-\mu), \qquad t > 0, \qquad \mu < 0. \end{aligned} \tag{1.2}$$

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Assuming the process to be steady-state, we separate the time variable, setting

$$Y(t,x,\mu) = \Psi(x,\mu)\,e^{i\omega t}. \tag{1.3}$$

Substituting (1.3) into (1.1), we reduce the unsteady-state boundary-value problem to the steady-state one

$$\left(i\omega + \mu\frac{\partial}{\partial x} + 1\right)\Psi(x,\mu) = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} \exp(-\mu'^2)\Psi(x,\mu')\,d\mu'. \tag{1.4}$$

The boundary conditions (1.2) are transformed to the form

$$\Psi(0,\mu) = \Psi(0,-\mu) + 2U\mu, \quad \mu > 0, \quad \Psi(d,\mu) = \Psi(d,-\mu), \quad \mu < 0. \tag{1.5}$$

In what follows, we consider the boundary-value problem which is to solve Eq. (1.4) with boundary conditions (1.5).

2. Characteristic System of Equations. Eigenfunctions. We divide the variables in Eq. (1.4) as follows:

$$\Psi_\eta(x,\mu) = \exp\left[-\frac{x}{\eta}(1+i\omega)\right]\Phi_1(\eta,\mu) + \exp\left[-\frac{d-x}{\eta}(1+i\omega)\right]\Phi_2(\eta,\mu), \tag{2.1}$$

where $\eta \in \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and $\mu > 0$. Substituting (2.1) into (1.4) and accepting the normalization condition

$$(1+i\omega)n_k(\eta) = \int\limits_{-\infty}^{\infty} \exp(-\mu^2)\Phi_k(\eta,\mu)\,d\mu \qquad (k = 1, 2), \tag{2.2}$$

we obtain the following characteristic system:

$$(\eta-\mu)\Phi_1(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta n_1(\eta), \qquad (\eta+\mu)\Phi_2(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta n_2(\eta). \tag{2.3}$$

The solution of system (2.3) depends considerably on whether or not the spectral parameter belongs to the real axis. We consider two cases.

(1) Let $\eta \notin \mathbb{R}$. The eigenfunctions are of the form

$$\Phi_1(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta\frac{1}{\eta-\mu}n_1(\eta), \qquad \Phi_2(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta\frac{1}{\eta+\mu}n_2(\eta). \tag{2.4}$$

Substituting (2.4) into (2.2), we obtain conditions that are imposed on the eigenfunctions of the discrete spectrum: $\Lambda(z;\omega) = 0$, where

$$\Lambda(z;\omega) = \lambda_c(z) + i\omega, \qquad \lambda_c(z) = 1 + z\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} \exp(-\mu^2)\frac{1}{\mu-z}\,d\mu. \tag{2.5}$$

The dispersion function (2.5) and its zeros and properties were studied by Latyshev and Yushkanov [12].

(2) Let $\eta \in \mathbb{R}$. We find the solution of system (2.3) in the class of generalized functions [13]:

$$\begin{aligned}\Phi_1(\eta,\mu) &= \frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta-\mu}n_1(\eta) + g_1(\eta)\delta(\eta-\mu),\\ \Phi_2(\eta,\mu) &= \frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta+\mu}n_2(\eta) + g_2(\eta)\delta(\eta+\mu).\end{aligned} \tag{2.6}$$

Here Px^{-1} means the distribution, i.e., the basic value of the Cauchy action integral, and $\delta(x)$ is the Dirac delta function.

After substitution of (2.6) into the normalization condition (2.2), one can find $g_{1,2}(\eta)$. Thus, system (2.6) is transformed to the form

$$\Phi_1(\eta,\mu) = \left[\frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta-\mu} + \exp(\eta^2)\Lambda(\eta;\omega)\delta(\eta-\mu)\right]n_1(\eta), \tag{2.7}$$

$$\Phi_2(\eta,\mu) = \Big[\frac{1}{\sqrt{\pi}}\eta P\,\frac{1}{\eta+\mu} + \exp(\eta^2)\Lambda(\eta;\omega)\delta(\eta+\mu)\Big] n_2(\eta).$$

Let

$$\Phi(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta P\,\frac{1}{\eta-\mu} + \exp(\eta^2)\Lambda(\eta;\omega)\delta(\eta-\mu). \tag{2.8}$$

Using the equality (2.8), we rewrite (2.7) as follows:

$$\Phi_1(\eta,\mu) = \Phi(\eta,\mu)n_1(\eta), \qquad \Phi_2(\eta,\mu) = \Phi(\eta,-\mu)n_2(\eta). \tag{2.9}$$

Thus, we have obtained the eigenfunctions of discrete (2.4) and continuous (2.9) spectra.

3. Expansion of the Boundary-Value Problem in Eigenvectors. We shall find the solution of problem (1.4) and (1.5) as the expansion in terms of the eigenfunctions of the characteristic system (2.3):

$$\Psi(x,\mu,\omega) = a_1(\eta_0;\omega)\Phi_1(\eta_0,\mu)\exp\Big[-\frac{x}{\eta_0}(1+i\omega)\Big]$$

$$+ a_2(\eta_0;\omega)\Phi_2(\eta_0,\mu)\exp\Big[-\frac{d-x}{\eta_0}(1+i\omega)\Big] + \int_0^\infty A_1(\eta;\omega)\exp\Big[-\frac{x}{\eta}(1+i\omega)\Big]\Phi_1(\eta,\mu)\,d\eta$$

$$+ \int_0^\infty A_2(\eta;\omega)\exp\Big[-\frac{d-x}{\eta}(1+i\omega)\Big]\Phi_2(\eta,\mu)\,d\eta, \tag{3.1}$$

where $\mathrm{Re}\,[(1+i\omega)/\eta_0] > 0$.

Substituting $x = 0$ and $x = d$ into (3.1) and making allowance for (2.8), we obtain

$$\Psi(0,\mu,\omega) = a_1(\eta_0;\omega)\Phi_1(\eta_0,\mu) + a_2(\eta_0;\omega)\Phi_2(\eta_0,\mu)\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]$$

$$+ \int_0^\infty A_1(\eta;\omega)\Phi(\eta,\mu)n_1(\eta)\,d\eta + \int_0^\infty A_2(\eta;\omega)\exp\Big[-\frac{d}{\eta}(1+i\omega)\Big]\Phi(\eta,-\mu)n_2(\eta)\,d\eta,$$

$$\Psi(d,\mu,\omega) = a_1(\eta_0;\omega)\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]\Phi_1(\eta_0,\mu) + a_2(\eta_0;\omega)\Phi_2(\eta_0,\mu)$$

$$+ \int_0^\infty A_1(\eta;\omega)\exp\Big[-\frac{d}{\eta}(1+i\omega)\Big]\Phi(\eta,\mu)n_1(\eta)\,d\eta + \int_0^\infty A_2(\eta;\omega)\Phi(\eta,-\mu)n_2(\eta)\,d\eta.$$

With the boundary conditions (1.5) taken into account, we have

$$2U\mu = \frac{1}{\sqrt{\pi}}\frac{2\eta_0\mu}{\eta_0^2-\mu^2}\Big\{a_1(\eta_0;\omega)n_1(\eta_0) - a_2(\eta_0;\omega)\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]n_2(\eta_0)\Big\}$$

$$+ \int_0^\infty A_1(\eta;\omega)n_1(\eta)\Big[\Phi(\eta,\mu) - \Phi(\eta,-\mu)\Big]\,d\eta$$

$$+ \int_0^\infty A_2(\eta;\omega)\exp\Big[-\frac{d}{\eta}(1+i\omega)\Big]\,n_2(\eta)[\Phi(\eta,-\mu) - \Phi(\eta,\mu)]\,d\eta; \tag{3.2}$$

$$0 = \frac{1}{\sqrt{\pi}}\frac{2\eta_0\mu}{\eta_0^2-\mu^2}\Big\{\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]a_1(\eta_0;\omega)n_1(\eta_0) - a_2(\eta_0;\omega)n_2(\eta_0)\Big\}$$

$$+ \int_0^\infty \exp\Big[-\frac{d}{\eta}(1+i\omega)\Big]A_1(\eta;\omega)n_1(\eta)[\Phi(\eta,\mu) - \Phi(\eta,-\mu)]\,d\eta$$

$$+\int_0^\infty A_2(\eta;\omega)n_2(\eta)[\Phi(\eta,-\mu)-\Phi(\eta,\mu)]\,d\eta. \tag{3.3}$$

We introduce the one-sided functions

$$n_k^+(\eta)=\begin{cases} n_k(\eta), & \eta\geqslant 0,\\ 0, & \eta<0,\end{cases} \qquad A_k^+(\eta;\omega)=\begin{cases} A_k(\eta;\omega), & \eta\geqslant 0,\\ 0, & \eta<0,\end{cases} \qquad k=1,\,2. \tag{3.4}$$

Using relations (3.2)-(3.4) and the fact that $\Phi(\eta,\mu)=\Phi(-\eta,-\mu)$, we write the system

$$\frac{1}{\sqrt{\pi}}\varphi(\mu;\omega)+\int_{-\infty}^{\infty}\Phi(\eta,\mu)n(\eta;\omega)\,d\eta=2U\mu, \quad \frac{1}{\sqrt{\pi}}\psi(\mu;\omega)+\int_{-\infty}^{\infty}\Phi(\eta,\mu)m(\eta;\omega)\,d\eta=0, \tag{3.5}$$

where

$$\begin{gathered}
\varphi(\mu;\omega)=\frac{2\eta_0\mu}{\eta_0^2-\mu^2}\Big[a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]n_2(\eta_0)\Big],\\
\psi(\mu;\omega)=\frac{2\eta_0\mu}{\eta_0^2-\mu^2}\Big[\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)n_2(\eta_0)\Big],\\
n(\eta;\omega)=A_1^+(\eta;\omega)n_1^+(\eta)-A_1^+(\omega;-\eta)n_1^+(-\eta)\\
+\exp\Big[\frac{d}{\eta}(1+i\omega)\Big]A_2^+(\omega;-\eta)n_2^+(-\eta)-\exp\Big[-\frac{d}{\eta}(1+i\omega)\Big]A_2^+(\eta;\omega)n_2^+(\eta),\\
m(\eta;\omega)=\exp\Big[-\frac{d}{\eta}(1+i\omega)\Big]A_1^+(\eta;\omega)n_1^+(\eta)\\
-\exp\Big[\frac{d}{\eta}(1+i\omega)\Big]A_1^+(\omega;-\eta)n_1^+(-\eta)+A_2^+(\omega;-\eta)n_2^+(-\eta)-A_2^+(\eta;\omega)n_2^+(\eta).
\end{gathered} \tag{3.5a}$$

We consider here and below that $\mu\in\mathbb{R}$, unless otherwise specified.

With allowance for (2.8), we transform system (3.5) into the following system of integral singular equations with the Cauchy kernel:

$$\frac{1}{\sqrt{\pi}}\varphi(\mu;\omega)+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\eta n(\eta;\omega)\frac{d\eta}{\eta-\mu}+\exp(\mu^2)\Lambda(\mu;\omega)n(\eta;\omega)=2U\mu,$$

$$\frac{1}{\sqrt{\pi}}\psi(\mu;\omega)+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\eta m(\eta;\omega)\frac{d\eta}{\eta-\mu}+\exp(\mu^2)\Lambda(\mu;\omega)m(\eta;\omega)=0.$$

Let us introduce the additive functions

$$N(z;\omega)=\int_{-\infty}^{\infty}\eta n(\eta;\omega)\frac{1}{\eta-z}\,d\eta. \tag{3.6}$$

$$M(z;\omega)=\int_{-\infty}^{\infty}\eta m(\eta;\omega)\frac{1}{\eta-z}\,d\eta. \tag{3.7}$$

The functions $N(z;\omega)$ and $M(z;\omega)$ are piecewise-analytic in the complex plane cut along the real axis. Using the Sokhotskii formulas [14] for values of the functions $N(z;\omega)$, $M(z;\omega)$, and $\Lambda(z;\omega)$, at the cut, we have

$$\Lambda^+(\mu;\omega)-\Lambda^-(\mu;\omega)=2\sqrt{\pi}i\mu\exp(-\mu^2); \tag{3.8}$$

$$N^+(\mu;\omega)-N^-(\mu;\omega)=2\pi i\mu n(\mu;\omega); \tag{3.9}$$

$$M^+(\mu;\omega)-M^-(\mu;\omega)=2\pi i\mu m(\mu;\omega). \tag{3.10}$$

Using (3.8)–(3.10), we write the system of Riemann scalar boundary-value problems

$$\Lambda^+(\mu;\omega)[\varphi(\mu;\omega)+N^+(\mu;\omega)-2\sqrt{\pi}U\mu]=\Lambda^-(\mu;\omega)[\varphi(\mu;\omega)+N^-(\mu;\omega)-2\sqrt{\pi}U\mu],$$
$$\Lambda^+(\mu;\omega)[\psi(\mu;\omega)+M^+(\mu;\omega)]=\Lambda^-(\mu;\omega)[\psi(\mu;\omega)+M^-(\mu;\omega)].$$

Let

$$P(z;\omega)=\Lambda(z;\omega)[\varphi(z;\omega)+N(z;\omega)-2\sqrt{\pi}Uz],\quad Q(z;\omega)=\Lambda(z;\omega)[\psi(z;\omega)+M(z;\omega)].$$

According to the analytical continuation theorem [14], the function $P(z;\omega)$ is analytic in the complex plane, except for the point at infinity at which this function has a first-order pole. According to the Liouville theorem [14], the function $P(z;\omega)$ is the first-order polynomial (c_0+c_1z). Taking into account that $P(0;\omega)=0$, we obtain $P(z;\omega)=c_1z$ and, hence,

$$N(z;\omega)=2\sqrt{\pi}Uz+\frac{c_1z}{\Lambda(z;\omega)}+\frac{2\eta_0z}{z^2-\eta_0^2}\Big\{a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]n_2(\eta_0)\Big\}. \tag{3.11}$$

Similarly, we find

$$M(z;\omega)=\frac{2\eta_0z}{z^2-\eta_0^2}\Big\{\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)n_2(\eta_0)\Big\}. \tag{3.12}$$

It is evident that (3.11) and (3.12) have first-order poles at the end points $\pm\eta_0$ and, in addition, (3.11) has a first-order pole at the point at infinity. The additive functions $M(z;\omega)$ and $N(z;\omega)$, which are specified by formulas (3.6) and (3.7) are, however, piecewise-analytic everywhere in the complex plane with a cut along the real axis. In view of this, to regard the solutions (3.11) and (3.12) as additive functions, it is necessary and sufficient to eliminate the previously found singularities.

With allowance for the behavior of the function $\Lambda(z;\omega)$ at infinity, we eliminate the pole at the point at infinity from the function $N(z;\omega)$ by setting $c_1=-2\sqrt{\pi}U\omega i$. It is easy to see that now $N(z;\omega)=O(1/z)$. It only remains for us to eliminate the first-order poles at the end points $\pm\eta_0$ in the functions $N(z;\omega)$ and $M(z;\omega)$. Making allowance for the evenness of the functions $\Lambda(z;\omega)$ and the oddness of the function $N(z;\omega)$, it is necessary and sufficient to eliminate the pole at the point η_0 and it will be eliminated at the point $-\eta_0$ automatically. Expanding $\Lambda(z;\omega)$ in a series in the vicinity of the point η_0 [note that the function $\Lambda(z;\omega)$ is analytic at the point η_0 and, therefore, we obtain the Taylor series] and taking into account the behavior of $M(z;\omega)$ in the neighborhood of this point, we obtain a system of two linear equations

$$\begin{gathered}a_1(\eta_0;\omega)n_1(\eta_0)-\exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]a_2(\eta_0;\omega)n_2(\eta_0)=-\frac{c_1}{\lambda_c'(\eta_0)},\\ \exp\Big[-\frac{d}{\eta_0}(1+i\omega)\Big]a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)n_2(\eta_0)=0.\end{gathered} \tag{3.13}$$

Solving system (3.13), we find the coefficients of the discrete spectrum:

$$\begin{gathered}a_1(\eta_0;\omega)=\frac{c_1}{\lambda_c'(\eta_0)n_1(\eta_0)(\exp[-2d(1+i\omega)/\eta_0]-1)},\\ a_2(\eta_0;\omega)=\frac{c_1}{\lambda_c'(\eta_0)n_2(\eta_0)(\exp[-d(1+i\omega)/\eta_0]-\exp[d(1+i\omega)/\eta_0])}.\end{gathered} \tag{3.14}$$

Now we find the coefficients of the continuous spectrum. To do this, we use the Sokhotskii formulas and relations (3.8) and (3.9), take the difference between the boundary values of the functions $N(z;\omega)$, and, with allowance for the fact that $M(z;\omega)\equiv0$ (owing to the choice of the coefficients of discrete spectrum), obtain the system of linear equations

$$n(\mu;\omega)=\frac{c_1\mu\exp(-\mu^2)}{\sqrt{\pi}\Lambda^+(\mu;\omega)\Lambda^-(\mu;\omega)},\qquad m(\mu;\omega)=0.$$

Solving this system, we write the following relations for the coefficients of the continuous spectrum:

$$A_1^+(\eta;\omega) = \frac{c_1\mu \exp(-\mu^2)}{\sqrt{\pi}\Lambda^+(\mu;\omega)\Lambda^-(\mu;\omega)n_1(\eta)(1-\exp[-2d(1+i\omega)/\eta])},$$
$$A_2^+(\eta;\omega) = \frac{c_1\mu \exp(-\mu^2)}{\sqrt{\pi}\Lambda^+(\mu;\omega)\Lambda^-(\mu;\omega)n_2(\eta)(\exp[d(1+i\omega)/\eta]-\exp[-d(1+i\omega)/\eta])}. \tag{3.15}$$

Thus, all coefficients of expansion are found in explicit form and are given by relations (3.14) and (3.15). The fact that the expansion is the solution of the initial boundary-value problem is directly verified. The uniqueness of the solutions follows from the impossibility of the nontrivial expansion of the zero in the eigenvectors of the characteristic equations. Thus, the solution of the initial boundary-value problem as the expansion (3.1) is found.

4. Analysis of the Solutions Obtained. To analyze the results obtained, we consider four limiting cases.

(1) Let $\omega \ll 1$. We expand the dispersion function $\Lambda(z;\omega)$ into a Laurent series in the neighborhood of the point at infinity:

$$\Lambda(z;\omega) = -\frac{1}{2z^2} + i\omega + o\Big(\frac{1}{z^2}\Big), \qquad |z| \to \infty. \tag{4.1}$$

We find the eigenvalues of the discrete spectrum from relation (4.1):

$$\eta_0 = \pm\frac{i-1}{1\sqrt{\omega}}. \tag{4.2}$$

We thus obtain that the eigenvalues of the discrete spectrum tend to infinity at small values of ω.

(2) Let $d \to \infty$. Using the relations for the coefficients of expansion (3.14) and (3.15), we have

$$a_1(\eta_0;\omega) = \frac{2\sqrt{\pi}i\omega U}{\lambda_c'(\eta_0)n_1(\eta_0)}, \qquad a_2(\eta_0;\omega) = 0; \tag{4.3}$$

$$A_1^+(\eta;\omega) = -\frac{2Ui\omega\mu \exp(-\mu^2)}{\Lambda^+(\mu;\omega)\Lambda^-(\mu;\omega)n_1(\eta)}, \qquad A_2^+(\eta;\omega) = 0. \tag{4.4}$$

It follows from the above relations that the contribution of $\Phi_2(\eta,\mu)$ tends to zero for $d \to \infty$, and the expansion (3.1) takes the form

$$\Psi(x,\mu,\omega) = a_1(\eta_0;\omega)\Phi_1(\eta_0,\mu)\exp\Big[-\frac{x}{\eta_0}(1+i\omega)\Big] + \int_0^\infty A_1(\eta;\omega)\exp\Big[-\frac{x}{\eta}(1+i\omega)\Big]\Phi_1(\eta,\mu)\,d\eta,$$

where $a_1(\eta_0;\omega)$ and $A_1(\eta;\omega)$ are given by relations (4.3) and (4.4).

On the other hand, $a_1(\eta_0;\omega)$ and $A_1(\eta;\omega)$ can be obtained based on the following arguments. Note that for $d \to \infty$, the functions $\psi(\mu;\omega)$ and $m(\eta;\omega)$, which are specified by relations (3.5a) and are in the second equation of system (3.5), vanish. Thus, system (3.5) reduces to the integral singular equation with the Cauchy kernel. Using the theorem of orthogonality and fullness of eigenfunctions on the numerical axis [15], we find relations for the coefficients of expansion:

$$a_1(\eta_0;\omega) = \frac{2\sqrt{\pi}i\omega U}{\lambda_c'(\eta_0)n_1(\eta_0)}, \quad A_1^+(\eta;\omega) = \frac{2Ui\omega\mu \exp(-\mu^2)}{\Lambda^+(\mu;\omega)\Lambda^-(\mu;\omega)n_1(\eta)}. \tag{4.5}$$

Clearly, the relations for the coefficients from (4.3) and (4.4) completely coincide with relations (4.5) obtained on the basis of the previously proved relations.

(3) Let $|\eta_0| \ll d$, $\omega \ll 1$, and $1 \ll x$. In this case (outside the Knudsen layer), only the discrete mode remains. The continuous mode disappears because of the fast decay of the exponent. The expansion (3.1) is of the form

$$\Psi(x,\mu,\omega) = a_1(\eta_0;\omega)\Phi_1(\eta_0,\mu)\exp\Big[-\frac{x}{\eta_0}(1+i\omega)\Big].$$

Let

$$\delta_n = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp(-\mu^2) a_1(\eta_0;\omega) \Phi_1(\eta_0,\mu) \exp\left[-\frac{x}{\eta_0}(1+i\omega)\right] d\mu.$$

With allowance for (2.4) and (4.3), we obtain

$$\delta_n = -\frac{2iU\omega \exp\left[-\frac{x}{\eta_0}(1+i\omega)\right]}{\lambda_c'(\eta_0)} \eta_0 t(\eta_0).$$

Taking into account the behavior of the function $\lambda_c(z)$ for small values of ω and using the relations for η_0, we write for the relative concentration

$$\mathrm{Re}\delta_n = -\frac{U\exp[-x\sqrt{\omega}(1-\omega)]}{2\sqrt{\omega}}[(1-\omega)\sin(x\sqrt{\omega}(1+\omega)) - (1+\omega)\cos(x\sqrt{\omega}(1+\omega))].$$

(4) Let $1 \ll d \leqslant |\eta_0|$, $\omega \ll 1$, $1 \ll x$, and $(d-x) \gg 1$. The expansion (3.1) takes the form

$$\Psi(x,\mu,\omega) = a_1(\eta_0;\omega)\left\{\Phi_1(\eta_0,\mu)\exp\left[-\frac{x}{\eta_0}(1+i\omega)\right] + \alpha\Phi_2(\eta_0,\mu)\exp\left[-\frac{d-x}{\eta_0}(1+i\omega)\right]\right\},$$

where

$$\alpha = \frac{a_2(\eta_0;\omega)}{a_1(\eta_0;\omega)} = \frac{n_1(\eta_0)}{n_2(\eta_0)}\exp[-d(1+i\omega)/\eta_0].$$

It is clear that the coefficient α characterizes the reflected wave. In this case, δ_n is of the form

$$\delta_n = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp(-\mu^2) a_1(\eta_0;\omega) \Phi_1(\eta_0,\mu) \exp\left[-\frac{x}{\eta_0}(1+i\omega)\right] d\mu$$

$$+\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \exp(-\mu^2) a_2(\eta_0;\omega) \Phi_2(\eta_0,\mu) \exp\left[-\frac{d-x}{\eta_0}(1+i\omega)\right] d\mu.$$

With allowance for (2.4) and (4.3), we obtain

$$\delta_n = \frac{2iU\omega}{\lambda_c'(\eta_0)}\eta_0 t(\eta_0)\left\{\exp\left[-\frac{x}{\eta_0}(1+i\omega)\right] + \exp\left[-\frac{2d-x}{\eta_0}(1+i\omega)\right]\right\} \Big/ \left\{\exp\left[-\frac{2d}{\eta_0}(1+i\omega)\right]-1\right\}.$$

The relation for the relative concentration, hence, is of the form

$$\mathrm{Re}\delta_n = \frac{r_1 r_2}{(r_2 a_2 - 1)^2 + (r_2 b_2)^2}\left\{a_1 + r_2[(a_1 a_2 + b_1 b_2)(a_2 - 1) + (b_2 a_1 - a_2 b_1) b_2]\right\}.$$

Here

$$a_1 = \cos(x\sqrt{\omega}(1+\omega)); \quad b_1 = \sin(x\sqrt{\omega}(1+\omega)); \quad r_1 = \exp[-x\sqrt{\omega}(1-\omega)];$$
$$a_2 = \cos(2d\sqrt{\omega}(1+\omega)); \quad b_2 = \sin(2d\sqrt{\omega}(1+\omega)); \quad r_2 = \exp[2d\sqrt{\omega}(1-\omega)].$$

Thus, in the present paper, we have developed a method that allows one to find exact solutions of boundary-value problems in the layer for unsteady-state model kinetic equations with mirror boundary conditions, the lower plate which bounds the gas executing normal harmonic oscillations and the upper one being at rest. Separation of the variables leads to the characteristic system. We have found the eigenfunctions of discrete and continuous spectra of the characteristic system. The expansion of the solution of the initial boundary-value problem into eigenfunctions has been found as well. The desired coefficients of expansion have been obtained in explicit form.

The problems with boundary conditions (1.5) can find application in solving very diverse problems of the kinetic theory of gas and plasma, in the theory of neutron (or electron) transport, in theoretical astrophysics, etc.

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